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Comments on q-Algebras

Simón Codriansky[1]

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A critical study of some elementary aspects of q-algebras is presented. The results are: (i) the q-algebras are related to para-Bose (para-Fermi) algebras only when both reduce to the usual Bose (Fermi) case, (ii) after performing a linear transformation of the operators A and $A^{\dagger}$ that satisfy the q-algebra relation $AA^{\dagger} - qA^{\dagger}A = I$, a generalized version of Penney's theorem (in the sense that the new operators satisfy noncanonical commutation and anticommutation relations) is obtained, (iii) the spectrum of one of the Hamiltonians of the system is obtained from the correspondence principle, and (iv) a whole family of q-algebra Hamiltonians is exhibited. This family has the property that the noncanonical commutation relation is stable.

1. INTRODUCTION

1.1. The purpose of the present paper is to examine a number of statements that have appeared in the literature concerning the q-algebras. They are: the relation of q-algebras to para-Bose algebras (Section 3); the question of performing a linear transformation of the operators involved (Section 4); and some delicate points when the limit $q \to \pm 1$ is taken (Section 2). Besides these, a derivation of the spectrum of $H(B, D)$ [see equation (7b)] is presented using correspondence arguments as in Born and Jordan (1925). The question of which functions qualify to play the role of Hamiltonians is also answered under the requirement of stability of the noncanonical commutation relation (Section 5). Sections 1.2–1.6 contain the definitions and main results relevant for the subsequent discussion.

1.2. Arik and Coon (1976) and Kuryshkin (1980) introduced an algebra defined by [$(A, A^{\dagger})_q$ will be called a q-product]

$$(A, A^{\dagger})_q \equiv AA^{\dagger} - qA^{\dagger}A = I \tag{1}$$

[1]Centro de Física, IVIC, Caracas 1020-A, Venezuela, and Departamento de Matemáticas y Física, IPC-UPEL, Caracas 1021, Venezuela.

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where the operators A and $A^\dagger$ are Hermitian conjugate of one another, I is the identity, and $0<q<1$ (Arik and Coon, 1976) or $q\geq -1$ (Kuryshkin, 1980). In this note I will consider $q\geq -1$. The special values $q=+1, -1$ yield the usual Bose and Fermi commutation relations. The number operator corresponding to (1) is given by Kuryshkin (1980) as

$$n_q=(1/2)\log_{|q|}[I+(q-1)A^\dagger A]^2 \tag{2}$$

Remark 1.2.1. Although the number operator for operators that satisfy the usual Bose (and Fermi) commutation relation is well defined as $n_b=A^\dagger A$, the number operator n_q defined by (2) is easily seen to be ill defined for $|q|=1$.

1.3. The algebraic scheme defined by (1) and (2) has been called the q-algebra by the Jannussis group (Brodimas *et al.*, 1981). These authors studied the q-algebras using a bosonic realization of the operators A and $A^\dagger$; this means that A and $A^\dagger$ are written as functions of a pair of operators d and $d^\dagger$ that satisfy the usual Bose algebra: $[\![d, d^\dagger]\!]_-\equiv dd^\dagger-d^\dagger d=I$. Once the operators A and $A^\dagger$ are bosonized, it is concluded by Brodimas *et al.* (1981) that "with the help of the number operator $n_q,\ldots$, the following relations have been proved for para-Bose operators,"

$$[\![n_q, A]\!]_-=-A, \qquad [\![n_q, A^\dagger]\!]_-=A^\dagger \tag{3}$$

Remark 1.3.1. Relations (3) define, in fact, the number operator n_q. They are satisfied not only in the para-Bose case, but also in Bose, Fermi, para-Fermi, and more general cases in which A and $A^\dagger$ are annihilation and creation operators (Kuryshkin, 1988).

Remark 1.3.2. The validity of relations (3) does not depend on the fact that A and $A^\dagger$ are bosonized. When a bosonic realization of A and $A^\dagger$ is used they must be recovered so as to be sure that the bosonization has been done consistently.

Remark 1.3.3. The statement quoted above (Brodimas *et al.*, 1981) is shown below to be false because the operators A and $A^\dagger$ are not para-Bose operators unless $q=1$.

1.4. A para-Bose and a para-Fermi algebra are defined by the double commutation relation ($[\![c^\dagger, c]\!]_\pm=c^\dagger c\pm cc^\dagger$)

$$[\![[\![c^\dagger, c]\!]_\pm, c]\!]_-=-2c \tag{4}$$

where the operators c and $c^\dagger$ are Hermitian conjugate of one another; the plus sign is for the para-Bose and the minus sign for the para-Fermi algebra. The number operator N is defined as

$$N=(1/2)([\![c, c^\dagger]\!]_\pm \pm pI) \tag{5}$$

where $p \geq 1$ is an integer that defines the order of the algebra labeling the irreducible representations. This is done as follows: let $|0\rangle$ be the unique vacuum of the Fock space in which c and $c^\dagger$ act; then $cc^\dagger|0\rangle = p|0\rangle$. The special value $p=1$ reproduces the Bose (plus sign) and Fermi (minus sign) cases. From (4) and (5) it is evident that relations (3) are satisfied if $(A, A^\dagger)$ are identified with $(c, c^\dagger)$.

Remark 1.4.1. If relations (3) are valid, the operators A and $A^\dagger$ will not automatically satisfy (4) and (5). It is shown in Section 3 that this occurs only for $(q, p) = (1, 1)$ and $(-1, 1)$.

1.5. Bosonization of Fermi, para-Bose, and para-Fermi algebras has been reported in the literature as a Bose representation of Fermi operators (Kademova, 1970; Kademova and Kálnay, 1970; Kálnay, 1977; Naka, 1978; Garbaczewski, 1985) and in connection with vibronic coupling in diatomic molecules (Schmutz, 1980). Also bosonization of more abstract algebras has been presented under the name of B-para algebras (Gonzalez-Bernardo *et al.*, 1982, and references therein; Gonzalez-Bernardo, 1988). Since bosonization is such an elastic formalism, it is tempting to relate para-Bose and para-Fermi algebras with q-algebras through the bosonization formalism; this is evident from the quotation in Section 1.2 from Brodimas *et al.* (1981). However, as is proved in this note, though these algebraic schemes are amenable to bosonization, the hoped-for relation does not exist for $|q| \neq 1$ (see Section 3).

1.6. The simplest Hamiltonian introduced in para-Bose algebra is proportional to the number operator (5). Such a system has been studied under the name of the para-Bose harmonic oscillator (Mukunda *et al.*, 1981). In Jannussis *et al.* (1982, and references therein), and Siafarikas *et al.* (1983, and references therein), a Hamiltonian is introduced in the following way: A and $A^\dagger$ are expressed in terms of a pair of Hermitian operators B and D in the form

$$A = [\hbar(1+q)]^{-1/2}[(mw)^{1/2}B + i(mw)^{-1/2}D] \tag{6a}$$

$$A^\dagger = [\hbar(1+q)]^{-1/2}[(mw)^{1/2}B - i(mw)^{-1/2}D] \tag{6b}$$

where m and w are real constants and $i = \sqrt{-1}$. If (1) is to be satisfied, it is found that B and D satisfy the noncanonical commutation relation (a commutator or anticommutator will be called noncanonical whenever it equals an operator that is not proportional to the identity)

$$[\![B, D]\!]_- = i\hbar I + [2i(q-1)/w(q+1)]H(B, D) \tag{7a}$$

where

$$H(B, D) = D^2/2m + mw^2B^2/2 \tag{7b}$$

The function $H(B, D)$ is called the Hamiltonian of the system. The appearance of the second term in the right-hand side of (7a) justifies calling this commutation relation "noncanonical," while the form of $H(B, D)$ lends to the system the name "harmonic oscillator." In Section 5, I present a derivation of the spectrum of $H(B, D)$ using the correspondence arguments of Born and Jordan (1925). In the same section a whole family of Hamiltonians is exhibited requiring that the noncanonical commutation relation (7a) be stable. In Section 4 a study of general linear transformations of A and $A^{\dagger}$ is presented; it is found that a general version of Penney's (1965) theorem is valid.

2. BOSONIZATION OF q-ALGEBRAS. COMMENTS

This section begins with a brief summary of the bosonization of A and $A^{\dagger}$ as proposed in Brodimas *et al.* (1981). Then in a number of comments it is shown that the familiar Bose and Fermi cases, though included in (1) as particular cases, have to be handled carefully because on one hand they cannot be considered as the limits $q \to 1$ (Bose) and $q \to -1$ (Fermi) of the function $f_q(n)$ (see below) and on the other hand the number operator n_q [see (2)] does not yield the well-known number operator $n_b = A^{\dagger}A$ of the Bose and Fermi cases in the limit $|q| \to 1$.

2.1. The Bosonization of Brodimas *et al.* (1981)

The operators A and $A^{\dagger}$ are written in the form

$$A = f_q(n_b)d, \qquad A^{\dagger} = d^{\dagger}f_q(n_b) \tag{8}$$

where $n_b = d^{\dagger}d$ is the Bose number operator. The function $f_q(n_b)$ is introduced so as to ensure that (1) is fulfilled. The Fock space on which A and $A^{\dagger}$ act after being bosonized is spanned by kets $|n\rangle = C_n(d^{\dagger})^n|0\rangle$, where C_n is a normalization constant and $|0\rangle$ is the Bose vacuum ket which is annihilated by d. As is well known, $|n\rangle$ is an eigenket of n_b with eigenvalue n. Acting with the above expressions for A and $A^{\dagger}$ on $|n\rangle$, it is found from (1) that

$$f_q^2(n) = (q^{n+1} - 1)/(n+1)(q-1), \qquad n = 0, 1, 2, \ldots \tag{9}$$

If $q = 1$, $f_1^2(n) = 1$, so that $A = d$, $A^{\dagger} = d^{\dagger}$, that is, the operators A and $A^{\dagger}$ coincide with the original Bose operators.

If $q = -1$, A and $A^{\dagger}$ are Fermi operators; in this case

$$(n+1)f_{-1}^2 = [1 + (-1)^n]/2 \tag{10}$$

so that $(n+1)f_{-1}^2 = \cos^2(\pi n/2)$ [$\cos^{2k}(\pi n/2)$ with k integer would do equally well. Naka (1978) presents the case $k = 2$] and therefore

$$A = (n_b + 1)^{-1/2}\cos(\pi n_b/2)d, \qquad A^{\dagger} = d^{\dagger}(n_b + 1)^{-1/2}\cos(\pi n_b/2) \tag{11}$$

From (8) and (9) it is easily found that

$$A^{\dagger}|n\rangle = [\![n+1]\!]^{1/2}|n+1\rangle, \qquad A|n\rangle = [\![n]\!]^{1/2}|n-1\rangle \tag{12a}$$

$$A^{\dagger}A|n\rangle = [\![n]\!]|n\rangle, \qquad n_q|n\rangle = n|n\rangle \tag{12b}$$

where $[\![n]\!] = (q^n - 1)/(q-1)$. At this point the review of the bosonization of Brodimas *et al.* (1981) is ended. The results (12a) justify calling A and $A^{\dagger}$ annihilation and creation operators, respectively, of certain type of particles with the ket $|n\rangle$ having n such particles (Kuryshkin, 1980).

Comment 2.1.1. The general expression (9) for $f_q^2(n)$ will now be examined as a function of q in the limit $n \to \infty$. Consider first $|q| < 1$:

$$(n+1)f_q^2(n) \to (q-1)^{-1} \tag{13}$$

showing that $f_q^2(n) \to 0$ when $n \to \infty$.

If $q > 1, f_q^2(n) \to \infty$ when $n \to \infty$. This, together with (13) and $f_1^2(n) = 1$, show that, as a function of q, $f_q^2(n)$ has an essential singularity at $q = 1$. As a result, the Bose case is not obtained as the limit $q \to 1$ of $f_q^2(n)$. The same is true for $q \to -1$ because the Taylor series that leads to (13) leaves outside its convergence radius this value of q. The conclusion is that the Bose and Fermi commutation relations have to be handled separately as compared to other values of q, although in (1) they are included as particular cases.

Comment 2.1.2. It can be argued that in certain situations the limit $n \to \infty$ may be forbidden for $|q| \neq 1$. For this to happen a certain value of n, say m, should exist such that $A^{\dagger}|m\rangle = 0$. This means $q^{m+1} = 1$, so that q is an $(m+1)$th root of 1. Since $|q| \neq 1$, no solution exists. The only case for which a finite number of kets is found is $q = -1$, for which $A^2 = A^{\dagger 2} = 0$; this shows that the limit (13) cannot be taken for $q = -1$. Because of this, the Fermi case is separated from other values of q.

Comment 2.1.3. At this point I would like to remark that the definition (1) of a q-algebra does not include all possible q-products among the operators A and $A^{\dagger}$. To complete the multiplication table of the q-algebra, $(A, A)_q = (1-q)A^2$, $(A^{\dagger}, A^{\dagger})_q = (1-q)A^{\dagger 2}$, and $(A^{\dagger}, A)_q = A^{\dagger}A - qAA^{\dagger}$ have to be specified. Of course, from (1), $A^{\dagger}A - qAA^{\dagger} = (1-q)A^{\dagger}A - q$, but the other products cannot be obtained from (1) and have to be given independently. Once A and $A^{\dagger}$ are bosonized, the matrix elements of $(1-q)A^2$ and $(1-q)A^{\dagger 2}$ are fixed, but the result cannot be written in terms of A, $A^{\dagger}$, and I. As an example, operating with A^2 on the bosonic ket $|n\rangle$ gives

$$A^2|n\rangle = (q^{2n-1} - q^n - q^{n-1} + 1)^{1/2}/(q-1)|n-2\rangle$$

If $(A, A)_q = (1-q)A^2$ is required to vanish, the only possibilities are $q = 1$ (Bose) and $A^2 = 0$ (Fermi). Instead of completing the multiplication table, specifying that the operator $A^\dagger A$ is diagonal allows determination of all matrix elements of A and $A^\dagger$ (Kuryshkin, 1980).

Comment 2.1.4. From what has been said in Comment 2.1.2 it follows that for any $q \neq -1$ an infinite number of kets is present (Kuryshkin, 1980). Of course, imposing the condition $A^{\dagger n} = 0$ for a certain value of $n > 2$ leaves only n kets, but this is an additional requirement.

Other possibilities can be imagined, for instance,

$$(1-q)A^2 = A^m \tag{14}$$

with m a positive integer. If this kind of relation is considered together with (1), a bosonized version can be studied, but the eigenkets of the operator n_b are no longer useful because in this basis (14) is never satisfied (unless $m=2$). Instead, eigenkets of the operator A would be more appropriate (the q-coherent states); call them $|K\rangle$ with $A|K\rangle = K|K\rangle$, then the allowed values of K are $|1-q|^{-l/(m-2)} \exp[-2\pi i l/(m-2)]$, $l = 0, 1, \ldots, n-3$; the state space is of dimension m. The case of a finite dimension state space is considered in Kuryshkin (1980) in connection with the so-called q-algebras without interaction.

Comment 2.1.5. Consider (12b) together with the definition (2) of the number operator. If $|q| = 1$, the basis of the logarithm is unity and therefore it is not a function any more because any argument may have as image any real number with no definite rule to establish the correspondence (in particular, the whole set of real numbers can be made to correspond only to zero and one). This is an additional argument that reinforces the result that the Bose and Fermi cases must be handled separately.

3. RELATION OF q-ALGEBRAS TO PARA-BOSE AND PARA-FERMI ALGEBRAS

Para-Bose and para-Fermi algebras are defined by (4) and (5) (see Section 1.3). Replacing (5) in (4), relations (3) are obtained with n_q replaced by N. I adopt the point of view that (3) does not define a para-Bose or para-Fermi algebra if the number operator is an arbitrary function of the operators c and $c^\dagger$. Only if N is given by (5) is the algebra para-Bose or para-Fermi. This definition, adopted in the current literature, can (of course) be modified if there are compelling reasons to do so; at this stage of the study of para-Bose and para-Fermi algebras this does not seem to be the case.

I will now look for those values of q such that the operators A and $A^\dagger$, which in this section are identified with c and $c^\dagger$ that appear in (4) and (5), satisfy simultaneously (1) and (4). It is proved that this happens only for $q = \pm 1$.

3.1. Para-Bose Algebras and q-Algebras

Assume that A and $A^\dagger$ satisfy (1) and at the same time are para-Bose operators of order p. Then the following algebraic relations must be valid simultaneously:

$$AA^\dagger - qA^\dagger A = I, \qquad A^\dagger A^2 - A^2 A^\dagger = -2A \tag{15}$$

From (15) it follows that

$$(1-q^2)A^\dagger A^2 = (q-1)A \tag{16}$$

which is identically satisfied if $q=+1$. Consider now $q \neq 1$ and operate with (16) on a ket $|n\rangle$; using (8) and (12), the result is

$$q^n + q^{n-1} = 2 \tag{17}$$

which should be valid for any $n \geq 2$ and fixed $q \neq 1$; notice that operating with (16) on $|0\rangle$ gives $0=0$, while operating on $|1\rangle$ gives $q=1$, which is not admissible because the case under consideration has $q \neq 1$. It is easily verified that no value of q ($\neq 1$) satisfies (17). The conclusion is that the operators A and $A^\dagger$ satisfy (1) and the para-Bose commutation relation only when both schemes reduce to the usual Bose case, i.e., $q=+1$, $p=1$. The value $p=1$ follows from (1) and (5) applied to $|0\rangle$ (see Section 1.4 for the definition of p).

The following is an alternative proof that q-algebras and para-Bose algebras are unrelated (unless $q=1, p=1$). To avoid confusion, I will return to the para-Bose operators c and $c^\dagger$ [as in (4)]. It is well known that the operators (Jordan, *et al.*, 1963) $J_0 = (1/4)[\![c, c^\dagger]\!]_+$, $J_1 = (i/4)(c^{\dagger 2} - c^2)$, and $J_2 = (1/4)(c^{\dagger 2} + c^2)$ satisfy

$$[\![J_2, J_0]\!]_- = iJ_1, \qquad [\![J_0, J_1]\!]_- = iJ_2, \qquad [\![J_1, J_2]\!]_- = -iJ_0 \tag{18}$$

If I ask the question, is it possible to write J_0, J_1, and J_2 as functions of A and $A^\dagger$ that satisfy (1)?, the answer is no.

Proof. Consider the basis of eigenstates of the operator $\mathbb{Q} = A^\dagger A$ (in particular, this basis can be the bosonic kets that have been used up to now) where A and $A^\dagger$ satisfy the q-algebra relation (1). Let the vacuum be $|0\rangle$. Write J_0, J_1, and J_2 as

$$J_0 = d_0^0 I + d_1^0 A + d_2^0 A^\dagger + d_3^0 A^2 + d_4^0 A^{\dagger 2} + d_5^0 A^\dagger A$$

$$J_1 = d_0^1 I + d_1^1 A + d_2^1 A^\dagger + d_3^1 A^2 + d_4^1 A^{\dagger 2} + d_5^1 A^\dagger A$$

$$J_2 = d_0^2 I + d_1^2 A + d_2^2 A^\dagger + d_3^2 A^2 + d_4^2 A^{\dagger 2} + d_5^2 A^\dagger A$$

where d_j^i, $i=0,1,2, j=1,\ldots,5$, are coefficients to be determined from the commutators (18). Replace the expansions for J_i, $i=0,1,2$, in the three commutators and take matrix elements in the chosen basis. It is shown after a tedious but straightforward algebra that the set of coefficients is identically zero. ■

3.2. Para-Fermi Algebras and q-Algebras

Assume that A and $A^\dagger$ satisfy (1) and at the same time are para-Fermi operators of order p. Then the following relations must be simultaneously valid:

$$AA^\dagger - qA^\dagger A = I, \qquad A^\dagger A^2 - 2AA^\dagger A + A^2 A^\dagger = -2A \tag{19}$$

From (19) it follows that

$$(q-1)^2 A^\dagger A^2 = -(1+q)A \tag{20}$$

which is satisfied without violating (1) if $q=-1$ and $A^2=0$, that is, A and $A^\dagger$ are Fermi operators. The value $q=+1$ is explicitly excluded because in this case $A=0$ and (1) is violated.

If $|q| \neq 1$, acting with (20) on $|n\rangle$ and using (8) and (12) gives the result

$$q^n - q^{n-1} = -2 \tag{21}$$

which should be valid for any $n \geq 2$ and $|q| \neq 1$. It is easily verified that no value of q ($\neq 1$) satisfies (21). Therefore, the operators A and $A^\dagger$ satisfy (1) and are para-Fermi operators only for $q=-1$, $p=1$, i.e., for the Fermi commutation relations. As before, $p=1$ follows from (1) and (5) applied to $|0\rangle$.

3.3. An Explicit Example

Consider the operators (Jackson, 1951; Arik and Coon, 1976; Jannussis *et al.*, 1983) D_q and T which act on a polynomial $P(x)$ of a real variable x as

$$D_q P(x) = [P(qx) - P(x)]/(q-1)x, \qquad TP(x) = xP(x) \tag{22}$$

It is easily verified that these operators satisfy the algebraic relation (1), $D_q T - qTD_q = I$. The operator D_q is related to $D = d/dx$ by (Jackson, 1951; Jannussis *et al.*, 1983)

$$D_q = (q^{xD} - 1)/(q-1)x \tag{23}$$

so that both xD and xD_q have x^m as eigenfunction with eigenvalues m and $(q^m - 1)/(q-1)$, respectively.

If D_q and T are required to satisfy the para-Bose algebraic relation, it follows that

$$[\![[\![T, D_q]\!]_+, T]\!]_- x^m = q^m (q+1) x^{m+1} \tag{24}$$

which equals $2x^{m+1}$ only if $q=1$.

4. LINEAR TRANSFORMATIONS OF A AND $A^\dagger$

In equations (6a), (6b) the transformation of A and $A^\dagger$ in terms of Hermitian operators B and D is presented. As a result the noncanonical commutation relation (7a) is obtained with $H(B, D)$ as in (7b). In this section, I tackle the question of performing a general linear transformation (in fact, a Bogoliubov transformation) of A and $A^\dagger$ [see (25) and (38) below] with the purpose of determining all product relations satisfied by the operators involved [B and D in (25); B and $B^\dagger$ in (38)]. Particular emphasis is placed on looking for those situations in which the product relation is a commutator or an anticommutator.

The main results of this section are:

1. If A and $A^\dagger$ are written in terms of two Hermitian operators B and D, then (a) B and D do not satisfy an anticommutation relation (noncanonical); (b) B and D satisfy a commutation relation (noncanonical) if $q \neq -1$.
2. If A and $A^\dagger$ are written in terms of one operator B and it Hermitian conjugate $B^\dagger$, then (a) B and $B^\dagger$ do not satisfy a commutation relation (noncanonical); (b) B and $B^\dagger$ satisfy an anticommutation relation (noncanonical) if $q \neq 1$ and $|a| = |b|$ (see Section 4.2).
3. Product relations that are neither a commutator nor an anticommutator are allowed in both cases for any value of q.

Two cases are considered: (a) B and D are Hermitian operators [a particular example is given by equations (6a), (6b)] and (b) $B = D^\dagger$.

4.1. Case (a)

A and $A^\dagger$ are written in the form

$$A = aB + bD, \qquad A^\dagger = a^*B + b^*D \tag{25}$$

where a and b are complex constants and the star indicates complex conjugation. The transformation (25) is assumed to be invertible. Replacement of these expressions in (1) gives

$$(BD + e^{if}DB) - qe^{if}(BD + e^{-if}DB) = (1/ab^*)I - [(1-q)/ab^*]H \tag{26}$$

where $H = |a|^2B^2 + |b|^2D^2$ and $e^{if} = a^*b/ab^*$.

4.1.1. If $e^{if} = -1$, $q \neq -1$, (26) reduces to

$$[\![B, D]\!]_- = [1/ab^*(1+q)]I - [(1-q)/(1+q)ab^*]H \tag{27}$$

If $q = -1$, B and D can be chosen so as to satisfy $[\![B, D]\!] = I$, but in this case $H(B, D) = I/2ab^*$. The particular values $a = [mw/\hbar(1+q)]^{1/2}$ and $b = i[\hbar(1+q)mw]^{-1/2}$ with m and w real bring (27) to

$$[\![B, D]\!]_- = i\hbar I + [2i(q-1)/w(q+1)](D^2/2m + mw^2B^2/2) \tag{28}$$

which is called the noncanonical harmonic oscillator in Jannussis *et al.* (1982) and Siafarikas *et al.* (1983). The function

$$H(B, D) = D^2/2m + mw^2B^2/2$$

is identified as the Hamiltonian of the system, as was mentioned in Section 1.5; I return to this in Section 5. If m and w are regarded simply as a pair of parameters not necessarily real, then (28) includes all cases considered in (27). A commutation relation similar to (28) is studied by Saavedra and Utreras (1981); see also Talukdar and Niyogi (1982).

Remark 4.1.1a. Since $e^{if} = -1$, the phases of a and b, fa and fb, respectively, are related by $fb = fa + (n+1/2)\pi$. From this $ab^* = |a||b|i(-1)^n$; replacing this in (27) and asking if the resulting equation

$$(1+q)i|a||b|(-1)^n(BD-DB) + (|a|^2B^2 + |b|^2D^2) = I \tag{27a}$$

can be factorized in the form $(rB+sD)(uB+vD) = I$ with u, r, u, and v constants, it is found that only for $q = 0$ is this factorization possible.

Remark 4.1.1b. Since $q \neq -1$, A and $A^\dagger$ are not Fermi operators. Therefore, the result (27) shows that Fermi operators are not related to noncanonical Bose operators or, since the transformation (25) is assumed to be invertible, noncanonical Bose operators cannot be expressed in terms of a finite number of Fermi operators. In a sense, this is a generalization of Penney's theorem (Penney, 1965) that Bose operators cannot be constructed by means of a finite number of Fermi operators. The relation to Penney's theorem is only "in a sense" because what Penney's theorem proves is that Bose creation and annihilation operators cannot be expressed in terms of a finite number of Fermi creation and annihilation operators. In Sections 4.1.1 and 4.1.2, B and D are not creation and annihilation operators.

4.1.2. If in (26) $e^{if} = +1$, the result is $(q \neq 1)$

$$[\![B, D]\!]_+ = [1/ab^*(1-q)]I - H/ab^* \tag{29}$$

From $e^{if} = +1$ the phases of a and b, fa and fb, respectively, are related by $fb = fa + n\pi$, with n an integer. As a consequence, $ab^* = |a||b|(-1)^n$, so from (29)

$$[|a|B + (-1)^n|b|D]^2 = I/(1-q) \tag{30}$$

and from here

$$|a|B = (-1)^{n+1}|b|D \pm I/(1-q)^{1/2} \tag{31}$$

The expressions for A and $A^\dagger$ that follow from (31) are

$$A = [b + (-1)^{n+1}\, e^{ifa}|b|]D \pm e^{ifa}I/(1-q)^{1/2} = \pm e^{ifa}I/(1-q)^{1/2} \tag{32a}$$

$$A^\dagger = [b^* + (-1)^{n+1}\, e^{-ifa}|b|]D \pm e^{-ifa}I/(1-q)^{1/2} = \pm e^{-ifa}I/(1-q)^{1/2} \tag{32b}$$

Relations (32a) and (32b) show that A and $A^\dagger$ are proportional to the identity. Operators such as (32a) and (32b) are trivial and are not considered in q-algebras (Kuryshkin, 1980). The conclusion is that nontrivial operators A and $A^\dagger$ cannot be represented as in (25) with B and D satisfying an anticommutation relation such as (29).

Remark 4.1.2a. Since $q \neq 1$, A and $A^\dagger$ are not Bose operators and thus Bose operators are not even considered as candidates to be written in terms of operators that satisfy an anticommutation relation (noncanonical). The result is actually much more general and states that for no value of q are A and $A^\dagger$ expressible as functions of B and D that satisfy (29).

4.1.3. Since for $e^{if} = +1$ the factorization (30) is possible for any value of q and since (27) factorizes only for $q = 0$, it is natural to look for other values of f and q such that (26) factorizes. Knowing f, one can relate the phases of a and b by

$$f_b = f_a + f/2 + n\pi \tag{33}$$

where $n = 0, 1, -1, 2, -2, \ldots$. From (33), $ab^* = |a||b|(-1)^e e^{-if/2}$, so that (26) reduces to

$$\begin{aligned}(-1)^n|a||b|(1+q^2-2q\cos f)^{1/2}(BD\,e^{-it} + DB\,e^{it})\\ +(1-q)(|a|^2B^2+|b|^2D^2) = I\end{aligned} \tag{34}$$

where $\tan t = (1+q)/(1-q)\tan f/2$. Equation (34) factorizes if

$$1+q^2-2q\cos f = (1-q)^2 \tag{35}$$

which implies

$$q(1-\cos f) = 0 \tag{36}$$

Therefore, only for $q = 0$ or $f = 2n\pi$ is factorization possible; these are exactly the two cases found above. If $q = 0$, the factorization reads

$$[|a|B + (-1)^n e^{if/2}|b|D][|a|B + (-1)^n e^{-if/2}|b|D] = I \tag{37}$$

4.2. Case (b)

A and $A^\dagger$ are written as linear combinations of one operator B and its Hermitian conjugate $B^\dagger$,

$$A = aB + bB^\dagger, \qquad A^\dagger = a^*B^\dagger + b^*B \tag{38}$$

where a and b are complex constants and the transformation (38) is assumed to be invertible. Replacement of (38) in (1) gives

$$BB^\dagger(|a|^2 - q|b|^2) + B^\dagger B(|b|^2 - q|a|^2) = I - (1-q)(ab^*B^2 + a^*bB^{\dagger 2}) \tag{39}$$

If $|a|^2 - q|b|^2 \neq 0$, (39) is a particular case of the generalized q-quantization (Kuryshkin, 1980). A commutator does not appear in (39) unless $q = +1$. On the other hand, an anticommutator appears if $|a| = |b|$ or $q = -1$. The result for $|a| = |b|$ is $(q \neq 1)$

$$[\![B, B^\dagger]\!]_+ = I/[|a|^2(1-q)] - (e^{i(fa-fb)}B^2 + e^{-i(fa-fb)}B^{\dagger 2}) \tag{40}$$

which factorizes as

$$(e^{ifa}B + e^{ifb}B^\dagger)(e^{-ifb}B + e^{-ifa}B^\dagger) = I/[|a|^2(1-q)] \tag{41}$$

Defining

$$U = |a|(1-q)^{1/2}(e^{ifa}B + e^{ifb}B^\dagger) = (1-q)^{1/2}A$$

we find that (41) implies

$$UU^\dagger = I = U^\dagger U \tag{42}$$

and the operator U is unitary for any value of $q \neq 1$. For $q = -1$ and $|a| \neq |b|$ a factorization similar to (41) is not obtained.

Remark 4.2.1. If $q = +1$, then A and $A^\dagger$ are Bose operators; the same is true for B and $B^\dagger$. If $q \neq 1$, a noncanonical anticommutator is possible, showing that a finite number of noncanonical Fermi operators do not form a Bose operator—Penney's theorem (Penney, 1956) again, but for the case of (noncanonical) anticommutator relations. In this case the relation to Penney's theorem is closer than in Section 4.1 because B and $B^\dagger$ can be thought of as annihilation and creation operators, respectively. (At this point I would like to thank A. J. Kálnay and C. A. Gonzalez-Bernardo for calling to my attention Penney's theorem.)

4.3. Equation (26) can be written in the form (for any value of q and of the constants a and b)

$$BD + e^{iG}DB = [I - (1-q)H]/ab^*(1 - q\,e^{if}) \tag{26a}$$

with $e^{iG} = (e^{if} - q)/(1 - q\,e^{if})$. The left-hand side of (26a) shows that the operators B and D obey an algebra whose product is defined as $(B, D)^G = BD + e^{iG}DB$; this product reduces to a commutator for any value of q if $e^{if} = -1$ and to an anticommutator if $e^{if} = +1$. For other values of e^{if} the product of the so-called intermediate statistics is obtained (Wilczek, 1982; Wu, 1984). In this sense it is apparent that the q-algebras include as particular cases a large group of algebras which, as has been shown in these notes, does not include para-Bose and para-Fermi algebra when $p \neq 1$.

When A and $A^\dagger$ are expanded as in Section 4.2 [see (38)] then B and $B^\dagger$ satisfy a q-algebra relation if $|a|^2 \neq q|b|^2$ and $|b|^2 \neq q|a|^2$.

4.4. As a conclusion of this section, I note that the operators A, $A^\dagger$ cannot be represented as a linear combination of two Hermitian operators [case (a)] that satisfy a noncanonical anticommutation relation or two operators that are Hermitian conjugate of one another [case (b)] that satisfy a noncanonical commutation relation. In the first case A and $A^\dagger$ are multiples of the identity, while in the second case A and $A^\dagger$ vanish. Conversely, the product $AA^\dagger - qA^\dagger A$ can be transformed into a commutator [case (a)] if $q \neq -1$ and the constants a and b satisfy $e^{if} = -1$ or an anticommutator [case (b)] if $q \neq 1$ and the constants a and b satisfy $|a| = |b|$. These results are a generalization of Penney's theorem in that noncanonical commutation or anticommutation relations are involved. Also, intermediate statistics are included in the q-algebras, as pointed out in Section 4.3.

5. HAMILTONIANS IN q-ALGEBRAS

As mentioned in the preceding section, the quadratic function $H(B, D) = D^2/2m + mw^2B^2/2$ defined in (7a) is identified in Jannussis *et al.* (1982) and Siafarikas *et al.* (1983) with the Hamiltonian of the system whose dynamical variables are B and D. In terms of A and $A^\dagger$, $H(B, D)$ has the simple form

$$H(B, D) = [\hbar w(q+1)/4][\![A, A^\dagger]\!]_+ \tag{43}$$

with eigenvalues [from (12)] (Jannussis *et al.* (1982); Siafarikas *et al.* (1983))

$$E_n = (q^{n+1} + q^n - 2)\hbar w(1+q)/4(q-1) \tag{44}$$

For $q = +1$, $E_n = \hbar w(n + 1/2)$.

The above procedure suggests the identification of $H_a = |a|^2B^2 + |b|^2D^2$ in case (a) and $H_b = ab^*B^2 + a^*bB^{\dagger 2}$ in case (b) with the Hamiltonian of the corresponding system. The disadvantage of such an attitude is that all cases in which A and $A^\dagger$ satisfy the q-algebra relation (1) correspond [through transformations (25) or (38)] to systems whose time evolution is governed by the same generic quadratic function, thus hinting that the q-algebras describe one and the same system (at least from a mathematical standpoint). However, the above will be true if $H(B, D)$ is the only function that can play the role of a Hamiltonian; this section is devoted to proving that this is not the case and to exhibit by explicit construction another such function. A second aim of this section is to rederive the spectrum (44) by imitating the procedure of Born and Jordan (1925) to obtain the harmonic oscillator spectrum and the matrix elements of B and D. The whole calculation will be performed using the function $H(B, D) = D^2/2m + mw^2B^2/2$. In what follows H is written instead of $H(B, D)$.

Remark 5.1. The system with Hamiltonian operator H is called the noncanonical harmonic oscillator in Jannussis *et al.* (1982) and Siafarikas *et al.* (1983). The reason for this name is the formal similarity of H with the Hamiltonian of the usual harmonic oscillator. At this point it is appropriate to ask whether a physical system is defined by a particular Hamiltonian together with a set of commutation relations or by a system of (in general coupled) differential equations. If the evolution equations are considered more basic than the Hamiltonian with the commutation relations—a point of view adopted in this note—then to a given set of differential equations may be associated more than one set of commutation relations, once a Hamiltonian is given. This point has been discussed by Wigner (1950), Yang (1951), Okubo (1980), and Palev (1982), among others. As far as the evolution equations are concerned, the equations obtained for B and D (see below) correspond to an anharmonic oscillator which is called noncanonical by Jannussis *et al.* (1982) and Siafarikas *et al.* (1983) due to the form of the commutator. Of course, it is possible to ask whether there exists any function $Z(B, D)$ that can play the role of a Hamiltonian and that generates the usual harmonic oscillator equations once the noncanonical commutation relations (7a) for B and D are taken into account. The answer is negative if the noncanonical commutator is to remain stable. Once the results of Section 5.1 are known, it is found that $Z(B, D)$ is nondiagonal in the basis that diagonalizes H. As a result, the commutator $[\![Z, H]\!]_-$ does not vanish and therefore, at least under the conditions of this note, a harmonic oscillator is not obtained. If the characteristics of the spectrum of the Hamiltonian are taken as a criterion that defines a physical system, then, as is evident from the spectrum (44) of H, the eigenvalues are not equally spaced as should be in the case of the usual harmonic oscillator. From what has been said in this Remark, a noncanonical harmonic oscillator is a new physical system that reduces to the usual harmonic oscillator when $q = 1$ and should not be considered as a generalization of it.

5.1. Rederivation of the Spectrum of $H(B, D)$

The time-evolution equations of B and D that satisfy (28) with the Hamiltonian (7b) are

$$\dot{B} = dB/dt = \{D + (K/\hbar w)[\![D, H]\!]_+\}/m \tag{45a}$$

$$\dot{D} = dD/dt = -mw^2\{B + (K/\hbar w)[\![B, H]\!]_+\} \tag{45b}$$

which for $q = +1$ reduce to the usual harmonic oscillator evolution equations $[K = (q-1)/(q+1)]$. The matrix elements of B and D satisfy, from (45a), (45b), and (28),

$$\dot{B}_{jk} = [D_{jk} + (K/\hbar w)(D_{jm}H_{mk} + H_{jm}D_{mk})]/m \tag{46}$$

$$\dot{D}_{jk} = -mw^2[B_{jk} + (K/\hbar w)(B_{jm}H_{mk} + H_{jm}B_{mk})] \tag{47}$$

$$B_{jm}D_{mk} - D_{jm}B_{mk} = i\hbar\delta_{jk} + (2iK/w)H_{jk} \tag{48}$$

Following the correspondence arguments of Born and Jordan (1925), write for the matrix elements of B, D, and H: $B_{jk}(t) = B_{jk}\, e^{2\pi i\nu(jk)t}$ and $D_{jk}(t) = D_{jk}\, e^{2\pi i\nu(jk)t}$, and H_{jk}, respectively. The coefficients B_{jk} and D_{jk} are to be determined from (46)-(48). Define the frequencies $\nu(jk)$ as usual by $h\nu(jk) = H_j - H_k$, where $H_j \equiv H_{jj}$ are the diagonal matrix elements of H. In order to satisfy (48) when $j \neq k$, H must be diagonal; in this basis, equations (46) and (47) reduce to

$$\dot{B}_{jk} = [1 + (K/\hbar w)(H_j + H_k)]D_{jk}/m \tag{49}$$

$$\dot{D}_{jk} = -mw^2[1 + (K/\hbar w)(H_j + H_k)]B_{jk} \tag{50}$$

and from here

$$2\pi\nu(jk) = \pm w[1 + (K/\hbar w)(H_j + H_k)] \tag{51}$$

The plus sign is for $j > k$ and the minus sign for $j < k$ and it has been assumed that the energy eigenvalues are ordered in such a way that $H_j > H_k$ if $j > k$. As a result of (51), there are at most two values of k for a given j ($k = j \pm 1$); therefore, the matrix elements of B and D vanish unless $k = j \pm 1$.

At this point the stage is set to derive the following results: if $k = j + 1$,

$$H_{j+1} = \hbar w(q+1)/2 + qH_j \tag{52}$$

$$H_j = mw^2(|B_{jj+1}|^2 + |B_{jj-1}|^2) \tag{53}$$

$$|B_{jj+1}|^2 = |B_{jj-1}|^2 + \hbar/2mw + KH_j/mw^2 \tag{54}$$

the minimum value of j is $j = 0$; from (53) and (54) it is found that

$$|B_{01}|^2 = \hbar(q+1)/4mw, \qquad H_0 = \hbar w(q+1)/4 \tag{55}$$

The result for H_0 coincides with (44) for $n = 0$. For arbitrary j, it is found that

$$|B_{jj+1}|^2 = [\hbar(q+1)/4mw] \sum_{r=0}^{j} q^r$$

$$H_j = [\hbar w(q+1)/4(q-1)](q^{j+1} + q^j - 2) \tag{56}$$

which completes the derivation.

Remark 5.1.1. The spectrum of $H(B, D)$ may also be obtained if instead of equations (45a), (45b) the evolution equations of the usual harmonic oscillator ($\dot{B} = D/m$, $\dot{D} = -mw^2 B$) are used together with the noncanonical commutation relation (28). This curious result, however, does not make sense, because the frequency condition for the usual harmonic oscillator is not fulfilled with the spectrum (44) [or (56)]. The same spectrum is obtained in both cases because a similar relation between the matrix elements of B and D holds, namely, $D_{j+1j} = imwB_{j+1j}$, and because the matrices that represent B and D have the same structure.

Remark 5.1.2. Once the results (56) are known, an operator $Z(B, D)$ can be determined so that the evolution equations for B and D are those of the usual harmonic oscillator. For this to happen $Z(B, D)$ must satisfy

$$[\![B, Z]\!]_- = -[(q-1)/mw(q+1)][\![D, H]\!]_+ \tag{57}$$

$$[\![D, Z]\!]_- = mw(q-1)/(q+1)[\![B, H]\!]_+ \tag{58}$$

and if the matrix elements of $Z(B, D)$ are computed, it is found that it is not diagonal and that its commutator with H does not vanish.

5.2. Other Hamiltonians

Any function $h(B, D)$ that plays the role of a Hamiltonian of the q-algebras will be required to commute with H. In this way the commutation relation (28) is stable. As a consequence of the results of Section 5.1, the function $h(B, D)$ has to be diagonal on the basis in which all the foregoing calculations were done. This is possible if $h(B, D)$ contains terms that involve only an even number of factors B and D if represented as a polynomial or a series expansion. A particular case is evidently H itself. As another example, consider the function (α, β, γ constants)

$$h(B, D) = \alpha B^4 + \beta D^4 + \gamma(BDBD + DBDB) \tag{59}$$

This function can be reshaped, using (28), as

$$\begin{aligned} h(B, D) = \alpha B^4 + \beta D^4 + \gamma(B^2D^2 + D^2B^2) + \hbar\gamma(\hbar + KH/w) \\ -(2iK\gamma/w)(BHD - DHB) \end{aligned} \tag{60}$$

This Hamiltonian has matrix elements that connect $j \to j+4$, $j+2$, j, $j-2$, $j-4$ only. In order that it be diagonal the matrix elements $j \to j+4$ and $j \to j+2$ must vanish; these conditions fix two of the coefficients α, β, and γ. The results are

$$\alpha = \beta m^4 w^4, \qquad \gamma = \beta m^2 w^2 \tag{61}$$

so that $h(B, D)$ reduces to

$$h(B, D) = \beta m^2[4H^2 + \hbar^2 w^2 + K\hbar wH - 2iKw(BHD - DHB)] \tag{62}$$

which is a function of H (the first three terms) plus a term proportional to $BHD - DHB$. To prove that this last term is not proportional to H, compute the (j, j) matrix element; the result is

$$(BHD - DHB)_{jj} = 2imw(H_{j+1j+1}|B_{jj+1}|^2 - H_{j-1j-1}|B_{j-1j}|^2) \tag{63}$$

which, after use of (56), gives

$$(BHD - DHB)_{jj} = [i\hbar(q+1)/2](H_{2j+1\,2j+1} + H_{2j\,2j} + H_{2j-1\,2j-1} + H_{jj} - H_{j-1\,j-1}) \quad (64)$$

which cannot be written as the (j, j) matrix element of a function of H. From this example, the general pattern to construct Hamiltonians is evident.

6. CONCLUDING REMARKS

A number of points concerning q-algebras and their relation to para-Bose and para-Fermi algebras have been clarified. Also, the results of performing a linear transformation in operator space have been explored; it was found that a generalization of Penney's theorem is possible. Finally, a family of q-algebra Hamiltonians has been exhibited by explicit construction of one of them and hinting that the way of constructing others follows the same pattern. Having at one's disposal a number of Hamiltonians helps to establish an eventual connection between q-algebras and physical systems. This is the main reason to explore the existence of q-algebra Hamiltonians other than $H(B, D)$.

As far as consistency is concerned, it is easy to check that once the matrix elements of B and D are known [see (56)] the results (12a), (12b) for the matrix elements of A and $A^{\dagger}$ are recovered. Also, it is found that the time evolution equations for A and $A^{\dagger}$ are

$$i\hbar\, dA/dt \equiv i\hbar\dot{A} = (A, H)_q = A\hbar w(q+1)/2 \quad (65)$$

$$-i\hbar\, dA^{\dagger}/dt \equiv -i\hbar\dot{A}^{\dagger} = (H, A^{\dagger})_q = A^{\dagger}\hbar(q+1)/2 \quad (66)$$

The equation for $\dot{A}^{\dagger}$ has this form, so that it coincides with the Hermitian conjugate of $\dot{A}$. It is also found that since H is a time-independent operator, $\dot{H} = 0$ does not coincide with $(H, H)_q = (1-q)H^2$. This is a situation similar to the one found when classical mechanics is expressed in terms of symmetric brackets; in fact, in this case the time evolution equation is obtained from the symmetric brackets only for the coordinate and momentum, but not for an arbitrary function of these dynamical variables (see Franke and Kálnay, 1970).

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